Minimum rank of a random graph over the binary field

Jisu Jeong[1] Choongbum Lee[2]
Po-Shen Loh[3] Sang-il Oum[1]

[1]KAIST

[2]Massachusetts Institute of Technology

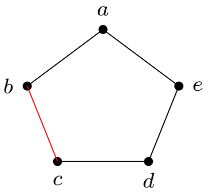
[3]Carnegie Mellon University

2013 KMS Annual Meeting

Definition (The minimum rank of a graph over a field)

A matrix M represents a graph G if

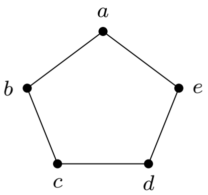
$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{pmatrix} & 1 & 0 & 0 & 1 \\ 1 & & 1 & 0 & 0 \\ 0 & 1 & & 1 & 0 \\ 0 & 0 & 1 & & 1 \\ 1 & 0 & 0 & 1 & \end{pmatrix} \end{array}
$$



There are many matrices that represent a graph.
Denote $\mathrm{mr}(\mathbb{F}, G)$.

Example $(\mathrm{mr}(\mathbb{F}_2, C_5) = 3)$

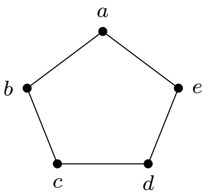
$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$



Thus, $\mathrm{mr}(\mathbb{F}_2, C_5) \leq 3$.

Example $(\mathrm{mr}(\mathbb{F}_2, C_5) = 3)$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} & 1 & 0 & 0 & 1 \\ 1 & & 1 & 0 & 0 \\ 0 & 1 & & 1 & 0 \\ 0 & 0 & 1 & & 1 \\ 1 & 0 & 0 & 1 & \end{pmatrix} \end{array}$$



Thus, $\mathrm{mr}(\mathbb{F}_2, C_5) \geq 3$.

Motivation

an eigenvalue λ

of a matrix A which represents a graph G

Motivation

the (geometric) multiplicity
of an eigenvalue λ

Motivation

the (geometric) multiplicity
of an eigenvalue λ
$= \text{nullity}(A - \lambda I)$

Motivation

the maximum multiplicity of an eigenvalue λ

$$= \max \operatorname{nullity}(A - \lambda I)$$

Motivation

$$
\begin{aligned}
&\max \text{ multiplicity of } \lambda \\
&= \max \operatorname{nullity}(A - \lambda I) \\
&= |V(G)| - \min \operatorname{rank}(A - \lambda I) \\
&= |V(G)| - \operatorname{mr}(G) \ (\because A - \lambda I \text{ represents } G)
\end{aligned}
$$

Thus,

$$\operatorname{mr}(G) = |V(G)| - \max \text{ multiplicity of } \lambda$$

Some properties

Some properties

- The miminum rank of G is at most 1 if and only if G can be expressed as the union of a clique and an independent set.
- A path P is the only graph of minimum rank $|V(P)| - 1$.
- For a cycle C, $\mathrm{mr}(C) = |V(C)| - 2$.
- If G' is an induced subgraph of G, then $\mathrm{mr}(G') \leq \mathrm{mr}(G)$.

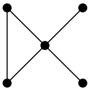
Known results

Theorem(Barrett, van der Holst, and Loewy, 2004)

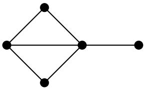
Let G be a graph. Then, $\mathrm{mr}(\mathbb{R}, G) \leq 2$ if and only if G is $(P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, K_{3,3,3})$-free.

Theorem(Hogben and van der Holst, 2006)

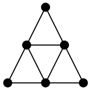
Let G be a 2-connected graph. Then, $\mathrm{mr}(\mathbb{R}, G) = n - 2$ if and only if G has no K_4-, $K_{2,3}$-, or T_3-minor.



$\ltimes$



dart



T_3

Known results

Theorem(Ding and Kotlov, 2006)

If $\mathbb{F}$ is a finite field, then for every k, the set of graphs of mininum rank at most k is characterized by finitely many forbidden induced subgraphs, each on at most $\left(\frac{|\mathbb{F}|^k}{2}+1\right)^2$ vertices.

Remark

- $\mathrm{mr}(\mathbb{F}_2, K_{3,3,3}) = 2$
- $\mathrm{mr}(\mathbb{R}, K_{3,3,3}) = 3$

Random graph

We consider the Erdős-Rényi random graph $G(n, p)$.
The vertex set of a random graph $G(n, p)$ is $\{1, 2, \cdots, n\}$ and two vertices are adjacent with probability p independently at random.
Given a graph property $\mathcal{P}$, we say that $G(n, p)$ possesses $\mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G(n, p)$ possesses $\mathcal{P}$ converges to 1 as n goes to infinity.

Known results

The minimum rank of a random graph over a field.

	$\mathbb{R}$ †	$\mathbb{F}_2$ ‡
$G(n, 1/2)$	$0.147n < \mathrm{mr} < 0.5n$	$n - \sqrt{2n} \leq \mathrm{mr}$
$G(n, p)$	$cn < \mathrm{mr} < dn$	

† Hall, Hogben, Martin, and Shader, 2010
‡ Friedland and Loewy, 2010

Known results

The minimum rank of a random graph over a field.

	$\mathbb{R}^\dagger$	$\mathbb{F}_2{}^\ddagger$
$G(n,1/2)$	$0.147n < \mathrm{mr} < 0.5n$	$n - \sqrt{2n} \leq \mathrm{mr}$
$G(n,p)$	$cn < \mathrm{mr} < dn$	!!!

† Hall, Hogben, Martin, and Shader, 2010
‡ Friedland and Loewy, 2010

Our results

Let $p(n)$ be a function s.t. $0 < p(n) \leq \frac{1}{2}$ and $np(n)$ is increasing. We prove that the minimum rank of $G(n, 1/2)$ and $G(n, p(n))$ over the binary field is at least $n - o(n)$ a.a.s.
We have two different proofs.

Theorem (using the 1st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.01$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.483\sqrt{n/p(n)}$ a.a.s. $(\sqrt{2 \ln 3})$

Theorem (using the 2st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - 1.415\sqrt{n}$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.178\sqrt{n/p(n)}$ a.a.s. $(\sqrt{2 \ln 2})$

Our results

Let $p(n)$ be a function s.t. $0 < p(n) \le \frac{1}{2}$ and $np(n)$ is increasing. We prove that the minimum rank of $G(n, 1/2)$ and $G(n, p(n))$ over the binary field is at least $n - o(n)$ a.a.s.
We have two different proofs.

Theorem (using the 1st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \ge n - \sqrt{2n} - 1.01$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \ge n - 1.483\sqrt{n/p(n)}$ a.a.s.

Theorem (using the 2st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \ge n - 1.415\sqrt{n}$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \ge n - 1.178\sqrt{n/p(n)}$ a.a.s.

Our results

Theorem (J., C.Lee, P.Loh, S.Oum, 2013+)

Let $p(n)$ be a function s.t. $0 < p(n) \leq \frac{1}{2}$ and $np(n)$ is increasing.

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.01$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.178\sqrt{n/p(n)}$ a.a.s.

Future work

	$\mathbb{R}$	$\mathbb{F}_2$
$G(n, 1/2)$	$0.147n < \mathrm{mr} < 0.5n$	$n - \sqrt{2n} \le \mathrm{mr}$
$G(n, p)$	$cn < \mathrm{mr} < dn$ (p fixed)	$n - 1.178\sqrt{n/p(n)} \le \mathrm{mr}$

- A nontrivial upper bound of the minimum rank of a random graph over the binary field is an open question.
- The minimum rank of a random graph over the other fields is unknown.
- The minimum rank of a random graph $G(n, p)$ is unknown.
- Is the minimum rank problem NP-complete??

Thank you.

Our results

Let $p(n)$ be a function s.t. $0 < p(n) \leq \frac{1}{2}$ and $np(n)$ is increasing. We prove that the minimum rank of $G(n, 1/2)$ and $G(n, p(n))$ over the binary field is at least $n - o(n)$ a.a.s.

We have two different proofs.

Theorem (using the 1st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.01$ a.a.s. (Proof)
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.483\sqrt{n/p(n)}$ a.a.s.

Theorem (using the 2st method)

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - 1.415\sqrt{n}$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.178\sqrt{n/p(n)}$ a.a.s.

Theorem

Let $\mathbb{F}_2$ be the binary field and $G\left(n, \frac{1}{2}\right)$ be a random graph. Then,

$$\mathrm{mr}\left(\mathbb{F}_2, G\left(n, \frac{1}{2}\right)\right) \geq n - \sqrt{2n} - 1.01$$

asymptotically almost surely.

Sketch of the proof.

$G = G(n, 1/2)$

$\mathcal{G}_n$: a set of all graphs with a vertex set $\{1, 2, \cdots, n\}$

$S_n(\mathbb{F}_2)$: a set of all $n \times n$ symmetric matrices over the binary field

There can be many different matrices representing the same graph. If one of them has rank less than r, then the minimum rank of this graph is less than r. Thus,

$$\sum_{\substack{\mathrm{mr}(\mathbb{F}_2,H)<r \\ H\in\mathcal{G}_2}} \mathbb{P}[G=H] \le \sum_{\substack{\mathrm{rank}(N)<r \\ N\in\mathcal{M}}} \mathbb{P}[G=G(N)].$$

Let M be an $n \times n$ random symmetric matrix s.t.
every entry on or above the main diagonal of M is 1 with $1/2$.
For $N \in S_n(\mathbb{F}_2)$, we have

$$\mathbb{P}[G=G(N)] = 2^n \mathbb{P}[M=N]$$

because the diagonal entries are decided with probability $1/2$ independently at random.

Therefore, we have

$$\begin{aligned}
\mathbb{P}[\mathrm{mr}(\mathbb{F}_2, G) < n - L] &= \sum_{\substack{\mathrm{mr}(\mathbb{F}_2,H)<n-L \\ H\in\mathcal{G}}} \mathbb{P}[G = H] \\
&\leq \sum_{\substack{\mathrm{rank}(N)<n-L \\ N\in\mathcal{M}}} \mathbb{P}[G = G(N)] \\
&= 2^n \sum_{\substack{\mathrm{rank}(N)<n-L \\ N\in\mathcal{M}}} \mathbb{P}[M = N] \\
&= 2^n \mathbb{P}[\mathrm{rank}(M) < n - L] \\
&= 2^n \mathbb{P}[\mathrm{nullity}(M) > L].
\end{aligned}$$

It is enough to show that $\mathbb{P}[\mathrm{nullity}(M) > \sqrt{2n} + 1.0]$ is $o(1/2^n)$.
So, we focus on $\mathbb{P}[\mathrm{nullity}(M) = L]$.

Lemma

Let M_i be an $i \times i$ random symmetric matrix such that every entry in the upper triangle and diagonal of M_i is 1 with probability $\frac{1}{2}$ independently at random. And let $P_{i,k}$ be the probability that M_i has nullity k. Then, $P_{1,0} = P_{1,1} = P_{2,0} = \frac{1}{2}$, $P_{2,1} = \frac{3}{8}$, $P_{2,2} = \frac{1}{8}$, $P_{i,-1} = 0$ for all i, $P_{i,k} = 0$ for all $i < k$, and

$$P_{i,k} = \frac{1}{2}P_{i-1,k} + \frac{1}{2^i}P_{i-1,k-1} + \frac{1}{2}(1 - \frac{1}{2^{i-1}})P_{i-2,k}$$

for $i \geq 3$, $k \geq 0$.